Circular coloring and circular flow in signed graphs

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Jaeger's circular flow conjecture

The 4-color theorem [AH76]

Every planar graph is 4-colorable.

[Tut66] Every cubic bridgeless planar graph admits a nowhere-zero 4-flow.

Tutte's 3-flow conjecture [Tut66]

Every 4-edge-connected graph admits a nowhere-zero 3-flow.

Jaeger's circular flow problem [Jae84]

Every $4k$-edge-connected graph admits a circular $\frac{2k+1}{k}$-flow.

- disproved for $k \geq 3$ [HLWZ18];
- verified for $6k$-edge-connectivity [LTWZ13].

Jaeger-Zhang Conjecture

Jaeger-Zhang Conjecture [Zha02]

Every planar graph of odd-girth at least $4k+1$ admits a circular $\frac{2k+1}{k}$-coloring.

- $k=1$: Grötzsch's theorem;
- $k=2$: verified for odd-girth 11 [DP17; CL20];
- $k=3$: verified for odd-girth 17 [CL20; PS22];
- $k \geq 4$:
  - verified for odd-girth $8k-3$ [Zhu01];
  - verified for odd-girth $\frac{20k-2}{3}$ [BKKW02];
  - verified for odd-girth $6k+1$ [LTWZ13].

Signed graphs

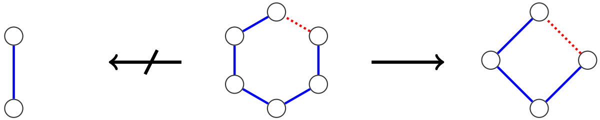
- A signed graph (G, σ) is a graph G together with an assignment $\sigma : E(G) \to \{+, -\}$.
- The sign of a closed walk (especially, a cycle) is the product of signs of all the edges of it.
- A switching at a vertex v is to switch the signs of all the edges incident to this vertex.

Theorem [Zas82]

Signed graphs (G, σ) and (G, σ') are switching equivalent if and only if they have the same set of negative cycles.

Homomorphisms of signed graphs

- We say (G, σ) admits a homomorphism to (H, π), write $(G, \sigma) \to (H, \pi)$, if there exists $\sigma' \equiv \sigma$ and a mapping $\varphi : V(G) \to V(H)$ that preserves the adjacencies and the signs of edges.



Circular $\frac{p}{q}$-coloring of signed graphs

Definition [NWZ21]

Given a positive even integer p and a positive integer q satisfying $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-coloring of (G, σ) is a mapping $\varphi : V(G) \to \{0, 1, ..., p-1\}$ such that for any positive edge uv,

$$q \leq |\varphi(u) - \varphi(v)| \leq p - q,$$

and for any negative edge uv,

$$|\varphi(u) - \varphi(v)| \leq \frac{p}{2} - q \quad \text{or} \quad |\varphi(u) - \varphi(v)| \geq \frac{p}{2} + q.$$

The circular chromatic number of (G, σ) is

$$\chi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-coloring}\}.$$

Signed circular cliques

Lemma [NWZ21]

A signed graph (G, σ) admits a circular $\frac{p}{q}$-coloring if and only if it admits an edge-sign preserving homomorphism to the signed circular clique $K^s_{p;q}$.

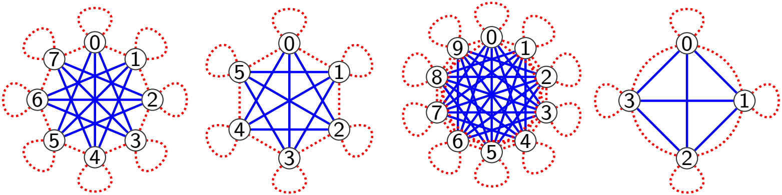


Figure: $K^s_{8;3} \prec K^s_{6;2} \prec K^s_{10;3} \prec K^s_{4;1}$

Circular $\frac{p}{q}$-flow in signed graphs

Definition [LNWZ22+]

Given a positive even integer p and a positive integer q where $q \leq \frac{p}{2}$, a circular $\frac{p}{q}$-flow in (G, σ) is a pair (D, f) where D is an orientation on G and $f : E(G) \to \mathbb{Z}$ satisfies the following conditions.

- For each positive edge e, $|f(e)| \in \{q, ..., p - q\}$;
- For each negative edge e, $|f(e)| \in \{0, ..., \frac{p}{2} - q\} \cup \{\frac{p}{2} + q, ..., p - 1\}$;
- For each vertex v of (G, σ), $\sum\limits_{(v,w)\in D} f(vw) = \sum\limits_{(u,v)\in D} f(uv)$.

The circular flow index of (G, σ) is

$$\Phi_c(G, \sigma) = \min\{\frac{p}{q} \mid (G, \sigma) \text{ admits a circular } \frac{p}{q}\text{-flow}\}.$$

Circular $\frac{p}{q}$-coloring and circular $\frac{p}{q}$-flow

Lemma [LNWZ22+]

Given a signed plane graph (G, σ) and its dual signed graph (G^*, σ^*),

$$\chi_c(G, \sigma) = \Phi_c(G^*, \sigma^*).$$

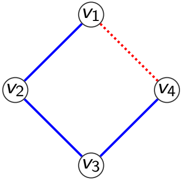


Figure: Circular $\frac{8}{3}$-coloring of C_{-4}

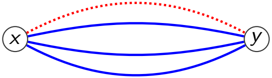


Figure: Circular $\frac{8}{3}$-flow in C^*_{-4}

Circular $\frac{2\ell}{\ell-1}$-coloring and circular $\frac{2\ell}{\ell-1}$-flow

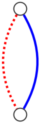


Figure: C_{-2}

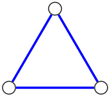


Figure: C_3

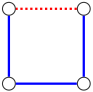


Figure: C_{-4}

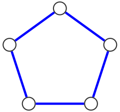


Figure: C_5

Let k be a positive integer.

- $\chi_c(G,+) \leq \frac{2k+1}{k}$ if and only if $(G,+) \to C_{2k+1}$.
- For bipartite G, $\chi_c(G,\sigma) \leq \frac{4k}{2k-1}$ if and only if $(G,\sigma) \to C_{-2k}$ [N**W**21+].

Homomorphisms of signed bipartite graphs

Proposition [HN90, NPW**22]**

For any integer k, a graph G is k-colorable if and only if $T_{k-2}(G)$ admits a homomorphism to C_{-k}.

4-color theorem restated

Given a planar graph G, the signed bipartite graph $T_2(G)$ admits a homomorphism to C_{-4}.

Lemma

- [NRS15] $G \to H \Leftrightarrow S(G) \to S(H)$;
- [N**W**Z21] Given a graph G, we have $\chi_c(S(G)) = 4 - \frac{4}{\chi_c(G)+1}$;
- [NRS15] $\chi(G) \leq k \Leftrightarrow S(G) \to (K_{k,k}, M)$ for $k \geq 3$.

Homomorphisms of signed bipartite graphs

4-color theorem restated [KNNW**21+, N**W**21+]**

Let G be a planar graph.

- $S(G) \to S(K_4)$;
- $S(G) \to \hat{B}^s_{16;5}$;
- $S(G) \to (K_{4,4}, M)$.

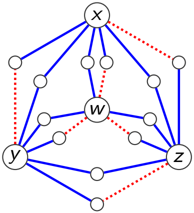


Figure: $S(K_4)$

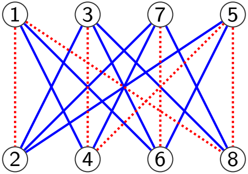


Figure: $\hat{B}^s_{16;5}$

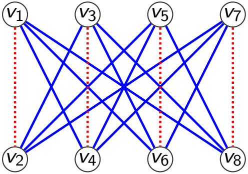


Figure: $(K_{4,4}, M)$

Signed bipartite analog of Jaeger-Zhang conjecture

Signed bipartite analog of Jaeger's circular flow conjecture

Every $g(k)$-edge-connected Eulerian signed graph admits a circular $\frac{4k}{2k-1}$-flow.

Signed bipartite analog of Jaeger-Zhang conjecture [NRS15]

Every signed bipartite planar graph of negative-girth at least $f(k)$ admits a homomorphism to C_{-2k}.

- $k = 2$: verified for negative-girth 8 (best possible) [NP**W**22];
- $k = 3, 4$: verified for negative-girth 14 and 20 [LS**W**W22+];
- $k \geq 5$:
  - verified for negative-girth $8k - 2$ [CNS20];
  - verified for negative-girth $6k - 2$ [LN**W**Z22+].

Thanks for your attention!